

A Compactness Theorem for Invariant Connections

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Necessary and sufficient conditions are given for the Palais-Smale Condition C to hold for the Yang-Mills functional for invariant connections on a principal bundle over a compact n -dimensional manifold. The connections are assumed to be invariant under the action of Lie group on the base manifold such that all orbits of the action have dimension greater than or equal to $n-3$.

§1. The compactness theorem

It is well-known that the Palais-Smale Condition C does not hold for the Yang-Mills functional on a principal bundle over a compact four-manifold. According to the Uhlenbeck weak compactness theorem, a Palais-Smale sequence will in general only have a subsequence that converges on the complement of a finite set of points. Moreover, even on the complement of these points, the convergence is not as good as one would desire. One only gets convergence with one derivative in L^2 . It is inconvenient to work with connections with one derivative in L^2 as there is no slice theorem for such connections.

It has been conjectured that Condition C holds if the Yang-Mills functional is restricted to connections that are invariant under a group action on the four-manifold, provided that all orbits of the action have dimension greater than or equal to one. The conjecture has been verified in a number of special cases; see Examples 2 and 3 in §2. In this paper we settle this conjecture for manifolds of any dimension and group actions all of whose orbits have codimension at most three.

Let X be a smooth compact n -dimensional Riemannian manifold, let G be a compact Lie group and let P be a smooth principal G -bundle over X . Let \mathcal{A} be the space of smooth connections on P and let \mathcal{G} be the group of smooth gauge transformations (bundle automorphisms) of P . Let

$$\mathcal{B} = \mathcal{A}/\mathcal{G}$$

be the space of gauge equivalence classes of smooth connections on P . We denote the gauge equivalence class of $A \in \mathcal{A}$ by $[A] \in \mathcal{B}$.

We choose a G -invariant positive definite inner product on the Lie algebra \mathfrak{g} ; if G is semisimple the negative of the Killing form provides a canonical invariant inner product. Then the Yang-Mills functional is given by $\mathcal{YM}[A] = \frac{1}{2} \int_X |F_A|^2 dx$ where F_A is the curvature of A . The critical points of the Yang-Mills functional are given by Yang-Mills equation $d_A^* F_A = 0$.

Next we let H be a compact Lie group with a smooth action ρ on X by isometries and with a lifted smooth action σ on P by bundle maps;

$$\begin{array}{ccc} H \times P & \xrightarrow{\sigma} & P \\ \downarrow & & \downarrow \\ H \times X & \xrightarrow{\rho} & X. \end{array}$$

The action of H on P induces actions of H on \mathcal{A} and \mathcal{G} . Let \mathcal{A}^H and \mathcal{G}^H be the fixed point sets of these actions. These are the space of H -invariant connections and the group of H -invariant gauge transformations. Let

$$\mathcal{E} = \mathcal{A}^H / \mathcal{G}^H.$$

We denote the equivalence class of $A \in \mathcal{A}^H$ by $[A, \sigma] \in \mathcal{E}$.

There is a natural map

$$\pi : \mathcal{E} \rightarrow \mathcal{B}$$

given by $\pi[A, \sigma] = [A]$. Note that this map is not in general injective. This occurs when H -invariant connections are gauge equivalent, but not gauge equivalent through H -invariant gauge transformation; see Example 1 in §2.

A sequence $A_n \in \mathcal{A}$ is said to be a Palais-Smale sequence if there exists $M > 0$ such that

$$\|F_{A_n}\|_{L^2(X, \Lambda^2 T^* X \otimes \text{Ad } P)} \leq M \quad \text{for all } n$$

and

$$\|d_{A_n}^* F_{A_n}\|_{L_{A_n}^{-1,2}(X, T^* X \otimes \text{Ad } P)} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Here $\|\cdot\|_{L_A^{-1,2}}$ is the norm on $L^{-1,2}$ dual to the norm $\|b\|_{L_A^{1,2}}^2 = \|b\|_{L^2}^2 + \|\nabla_A b\|_{L^2}^2$ on $L^{1,2}$. One says that Condition C holds if every Palais-Smale sequence has a subsequence that, after gauge transformations, converges to a Yang-Mills connection.

As we have mentioned, invariant connections can be gauge equivalent via non-invariant gauge transformations. Therefore there are two forms of Condition C for

the Yang-Mills functional for invariant connections: a weak form where we allow any gauge transformations and a strong form where we require the gauge transformations to be invariant. The weak form suffices for minimizing the Yang-Mills functional over the invariant connections; the strong form is needed for Morse theory on the space of invariant connections. Theorem 1 says that the weak form of Condition C always holds. Theorem 2 gives a necessary and sufficient condition for the strong form of Condition C to hold. Theorem 3 provides practical means of verifying this condition in many cases of interest.

Theorem 1. *If all orbits of the action of H on X have dimension $\geq n - 3$ and if $A_n \in \mathcal{A}^H$ is a Palais-Smale sequence, then there exists a subsequence, which we also denote A_n , and a sequence $g_n \in \mathcal{G}$ such that $g_n \cdot A_n$ converge strongly in $L_3^{1,2}$ to a Yang-Mills connection $A_\infty \in \mathcal{A}^H$ as $n \rightarrow \infty$.*

Theorem 2. *If all orbits of the action of H on X have dimension $\geq n - 3$, then the following two conditions are equivalent:*

- (1) *For any Palais-Smale sequence $A_n \in \mathcal{A}^H$ there exists a subsequence, which we also denote A_n , and a sequence $g_n \in \mathcal{G}^H$ such that $g_n \cdot A_n$ converge strongly in $L_3^{1,2}$ to a Yang-Mills connection $A_\infty \in \mathcal{A}^H$ as $n \rightarrow \infty$.*
- (2) *Any Yang-Mills connection in \mathcal{B} has only finitely many preimages in \mathcal{E} under the map π .*

Theorem 3. (a) *If G is semisimple, then any irreducible connection in \mathcal{B} has only finitely many preimages in \mathcal{E} under the map π . If furthermore H is connected or $Z(G) = 1$, then there exist at most one preimage.*

(b) *If H is semisimple, then any connection in \mathcal{B} has only finitely many preimages in \mathcal{E} under the map π .*

That a connection is irreducible here means that its isotropy subgroup is isomorphic to the center $Z(G)$ of G . The Morrey norm $L_3^{1,2}$ is discussed in §3. In two and three dimensions $L_3^{1,2} = L^{1,2}$. In four and more dimensions connections in $L_3^{1,2}$ have as good analytical properties as connections in $L^{1,2}$ in three dimensions.

Theorem 1 and the implication (2) \Rightarrow (1) in Theorem 2 are proven in §4. The implication (1) \Rightarrow (2) in Theorem 2 is proven in §6. Theorem 3 is a direct consequence of Proposition 6.2 and Lemma 5.5. Proposition 6.2 gives a detailed description of the preimages under π of elements of \mathcal{B} .

T. Parker, [21] Theorem 3.1, has given a proof of Condition (1) in Theorem 2 under the assumption that X is four-dimensional, the orbits have dimension greater than or

equal to one, G is semisimple and \mathcal{A}^H does not contain any reducible connections. However, U. Gritsch [11] has pointed out a gap in Parker's proof; in the proof of [21] Theorem 3.1 it is tacitly assumed, with no justification, that the gauge transformations given by [21] Theorem 2.3 are invariant. Parker's philosophy is to work exclusively with invariant functions and sections. Therefore it seems that to fill the gap one would need invariant good gauges.

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§2. Examples

Example 1 illustrates how invariant connections can be gauge equivalent through noninvariant gauge transformations. In examples 2 and 3 the weak form of Condition C is used to minimize the Yang-Mills functional on \mathcal{E} . In examples 4–6 the strong form of Condition C is used to do Morse theory for the Yang-Mills functional. Example 6 shows that one can do Morse theory even if the strong form of Condition C fails, provided that the extent of the failure can be controlled.

Example 1. Let $\mathrm{SO}(2)$ act on the trivial $\mathrm{U}(1)$ -bundle over S^1 by letting it act in the natural way on the base and trivially on the fibers. The gauge equivalence class of a connection is uniquely determined by its holonomy. Hence $\mathcal{B} = \mathrm{U}(1)$. The invariant connections are given by $\nabla + i\alpha d\theta$ with $\alpha \in \mathbb{R}$. Here ∇ denotes the trivial connection. Thus $\mathcal{A}^{\mathrm{SO}(2)} = \mathbb{R}$. The invariant gauge transformations are given by constants, so $\mathcal{G}^{\mathrm{SO}(2)} = \mathrm{U}(1)$. These act trivially on \mathcal{A} , so $\mathcal{E} = \mathcal{A}^{\mathrm{SO}(2)}/\mathcal{G}^{\mathrm{SO}(2)} = \mathcal{A}^{\mathrm{SO}(2)} = \mathbb{R}$. The connection $\nabla + i\alpha d\theta$ has holonomy $e^{-2\pi i\alpha}$. It follows that \mathcal{E} is the universal covering space of \mathcal{B} and π is the universal covering map. We also see that two invariant connections $\nabla + i\alpha d\theta$ and $\nabla + i\beta d\theta$ are gauge equivalent if and only if $\alpha - \beta$ is an integer. They are then related by the gauge transformation $e^{2\pi i(\alpha - \beta)\theta}$ which is not invariant unless $\alpha = \beta$.

Example 2. ([2], [21] §4, [23], [26]) The spin-2 representation $\mathrm{SO}(3) \times \mathbb{R}^5 \rightarrow \mathbb{R}^5$ induces an action of $\mathrm{SO}(3)$ on S^4 with the standard metric. The generic orbit of this action is a copy of $\mathrm{SO}(3)/(\mathbb{Z}_2 \oplus \mathbb{Z}_2)$. There are two exceptional orbits that are copies of $\mathrm{SO}(3)/\mathrm{O}(2) = \mathbb{R}P^2$. Let P_d be the principal $\mathrm{SU}(2)$ -bundle over S^4 with second Chern number d . The $\mathrm{SO}(3)$ -action on S^4 induces a $\mathrm{Spin}(3)$ -action on S^4 . The lifts of this action to P_d are classified by non-negative integral weights (w_-, w_+) such that

$d = (w_-^2 - w_+^2)/8$ and either $w_+ = w_- = 0$ or $w_+ \equiv w_- \equiv 1 \pmod{2}$. The Yang-Mills functional achieves its minimum on each corresponding space \mathcal{E} of invariant connections. This follows from Theorem 1, but was originally shown in [2] and [23] by dimensional reduction. If $w_- \geq 3$ and $w_+ \geq 3$, then the resulting Yang-Mills connection is non-(anti-)selfdual. If $(w_-, w_+) \neq (0, 0), (1, 1)$, then the resulting Yang-Mills connection is irreducible. This gives irreducible non-(anti-)selfdual Yang-Mills connections on P_d for all $d \neq \pm 1$.

This example gives the simplest known construction of non-(anti-)selfdual $SU(2)$ Yang-Mills connections over S^4 . The existence of such connections on P_0 is originally due to L. Sibner, R. Sibner and K. Uhlenbeck [24] using a different invariant minimization scheme. Whether there exist non-(anti-)selfdual Yang-Mills connections on P_1 and P_{-1} remains an open problem.

Example 3. ([2] §3 Remark 4, [26]) The standard representation $SO(3) \times \mathbb{C}^3 \rightarrow \mathbb{C}^3$ induces an isometric action of $SO(3)$ on $\mathbb{C}P^2$ with the Fubini-Study metric. The generic orbit is a copy of $SO(3)/\mathbb{Z}_2$. There are two exceptional orbits: a copy of $SO(3)/SO(2) = S^2$ and a copy of $SO(3)/O(2) = \mathbb{R}P^2$. Let P_d be the principal $SU(2)$ -bundle over $\mathbb{C}P^2$ with second Chern number d . The lifts of the induced $Spin(3)$ -action to P_d are classified by non-negative integral weights (w_-, w_+) such that $d = (w_-^2 - w_+^2)/4$ and either $w_- = 0$ and $w_+ \equiv 0 \pmod{2}$ or $w_- \equiv w_+ \equiv 1 \pmod{2}$. As in Example 2, the Yang-Mills functional achieves its minimum on each corresponding space \mathcal{E} of invariant connections. If $w_- \geq 3$ and $w_+ \geq 3$, then the resulting Yang-Mills connection is non-(anti-)selfdual. If $w_- \equiv w_+ \equiv 1 \pmod{2}$ and $(w_-, w_+) \neq (1, 1)$, then the resulting Yang-Mills connection is irreducible. This gives irreducible non-(anti-)selfdual Yang-Mills connections on P_d for all even $d \neq \pm 2$. This example is closely related to the previous one; $\mathbb{C}P^2$ is an $SO(3)$ -equivariant double cover of S^4 branched along one of the exceptional orbits.

Example 4. ([12]) There is a family M_{2g} of spin four-manifolds obtained as \mathbb{Z}_2 -quotients of the product of S^2 with a genus $2g$ surface. The natural action of $SO(3)$ on S^2 induces an action of $SO(3)$ on M_{2g} . The generic orbit of this action is a copy of $SO(3)/SO(2) \cong S^2$. There is a family of exceptional orbits parametrized by S^1 of orbit type $SO(3)/O(2) \cong \mathbb{R}P^2$. The action has a natural lift to the each chiral spinor bundle $S_{\pm}(M_{2g})$. There are no reducible connections on $S_{\pm}(M_{2g})$ so \mathcal{E} is a Hilbert manifold. By a min-max procedure for the Yang-Mills functional over \mathcal{E} , using Theorems 2 and 3, Gritsch shows that for generic $Spin(3)$ -invariant metrics on M_{2g} with $g \neq 1$ there exist irreducible non-(anti-)selfdual Yang-Mills connections on $S_{\pm}(M_{2g})$.

Example 5. ([13], [26], [27]) Let $SO(3)$ act diagonally on $S^2 \times S^2$ with a metric which is given by round metrics on each S^2 , possibly with different radii. The generic orbit of this action is a copy of $SO(3)$. There are two exceptional orbits that are copies

of S^2 ; these are given by pairs of identical points and pairs of antipodal points on S^2 . Let P_d be the principal $SU(2)$ -bundle over $S^2 \times S^2$ with second Chern number d . The lifts of the induced $Spin(3)$ -action are classified by non-negative integral weights (w_-, w_+) such that $w_+ \equiv w_- \pmod{2}$ and $d = (w_-^2 - w_+^2)/2$. There are reducible connections in \mathcal{E} so this space is not a Hilbert manifold. Thus one introduces the Hilbert manifold $\tilde{\mathcal{E}}$ of $Spin(3)$ -invariant connections with framings along an orbit. Let k be a nonnegative integer. In [13] we use equivariant Morse theory for a perturbed Yang-Mills functional on $\tilde{\mathcal{E}}$ with $(w_-, w_+) = (2k, 0)$ or $(0, 2k)$ to construct irreducible non-(anti-)selfdual Yang-Mills connections on P_d , with $d = \pm 2k^2$, of arbitrarily high energy. Such connections have also been obtained, for a different set of values of d , by H.-Y. Wang [27] using gluing techniques. The non-(anti-)selfdual Yang-Mills connections constructed in [26] Example 9.4 however are reducible.

Example 6. The Hopf fibration gives a free action of $SO(2)$ on S^3 . There is a unique principal $SU(2)$ -bundle over S^3 and the action has a unique lift to this bundle: the trivial bundle and the trivial lift. It follows from Proposition 6.2 that the trivial connection in \mathcal{B} has infinitely many preimages in \mathcal{E} under the map π . It then follows from Theorem 2 that the Yang-Mills functional on the space \mathcal{E} of invariant connections does not satisfy the strong form of Condition C. This is also shown in [15] by dimensional reduction to a Yang-Mills-Higgs type functional over S^2 .

However, any nontrivial Yang-Mills connection is irreducible, and has only finitely many preimages in \mathcal{E} by Theorem 3(a). Arguing as in the proof of Theorem 2, one can prove the following Lemma: *The strong form of Condition C holds for any Palais-Smale sequence that has energy bounded from below by a positive number.*

Using this one can still establish a Morse theory for a perturbed Yang-Mills functional on $\tilde{\mathcal{E}}$, the Hilbert manifold of invariant connections with invariant framings along an orbit. Construct a perturbed Yang-Mills functional on $\tilde{\mathcal{E}}$ that has isolated non-degenerate critical points and is equal to the Yang-Mills functional except near the degenerate critical points. Using the fact that the Yang-Mills functional on \mathcal{B} satisfies Condition C, one can show that the critical values of the perturbed Yang-Mills functional on $\tilde{\mathcal{E}}$ form a discrete set. One gets a decomposition of $\tilde{\mathcal{E}}$ as a CW-complex. Each critical point of the perturbed Yang-Mills functional contributes one cell. The critical value zero is attained at the preimages in $\tilde{\mathcal{E}}$ of the trivial connection. Each of these contributes one 0-cell. Thus there are infinitely many 0-cells. The Hilbert manifold $\tilde{\mathcal{E}}$ is connected, so there then have to be infinitely many 1-cells. However, it follows from the above Lemma that any positive critical value is attained at only finitely many critical points and therefore contributes only finitely many cells. We conclude that there exist $SU(2)$ Yang-Mills connections over S^3 of arbitrarily high energy.

§3. Analysis in Morrey spaces

In Theorems 1 and 2 we establish convergence in the Morrey norm $L_3^{1,2}$. There are two reasons for using this norm. First, an H -invariant function in $L^{1,2}(X)$ is automatically in $L_3^{1,2}(X)$ if all orbits of the action of H on X have dimension $\geq n-3$. Second, the space $L_3^{1,2}(X)$ has as good analytical properties as the space $L^{1,2}$ on a three-manifold, see Lemmas 3.2–3.5, in particular Lemma 3.3. In this section we review the basic properties of Morrey spaces.

The Morrey space $L_\lambda^p(\mathbb{R}^n)$, with $p \in [1, \infty)$ and $\lambda \in \mathbb{R}$, is defined as the space of all $f \in L^p(\mathbb{R}^n)$ such that

$$\sup_{\rho \in (0,1]} \sup_{x \in \mathbb{R}^n} \rho^{\lambda-n} \|f\|_{L^p(B_\rho(x))}^p < \infty.$$

It is a Banach space with norm

$$\|f\|_{L_\lambda^p(\mathbb{R}^n)}^p = \|f\|_{L^p(\mathbb{R}^n)}^p + \sup_{\rho \in (0,1]} \sup_{x \in \mathbb{R}^n} \rho^{\lambda-n} \|f\|_{L^p(B_\rho(x))}^p.$$

The Morrey space $L_\lambda^{k,p}(\mathbb{R}^n)$, with k a positive integer, $p \in [1, \infty)$ and $\lambda \in \mathbb{R}$, is defined as the space of all $f \in L^{k,p}(\mathbb{R}^n)$ such that $\partial^\alpha f \in L_\lambda^p(\mathbb{R}^n)$ for all α with $|\alpha| \leq k$. It is a Banach space with norm

$$(3.1) \quad \|f\|_{L_\lambda^{k,p}(\mathbb{R}^n)}^p = \sum_{|\alpha| \leq k} \|\partial^\alpha f\|_{L_\lambda^p(\mathbb{R}^n)}^p.$$

The Morrey space $f \in L_\lambda^{-k,p}(\mathbb{R}^n)$, with k a positive integer, $p \in [1, \infty)$ and $\lambda \in \mathbb{R}$, is defined as the space of all $f \in L^{-k,p}(\mathbb{R}^n)$ for which there exist $g_\alpha \in L_\lambda^p(\mathbb{R}^n)$, $|\alpha| \leq k$, such that

$$(3.2) \quad f = \sum_{|\alpha| \leq k} \partial^\alpha g_\alpha.$$

It is a Banach space with norm

$$(3.3) \quad \|f\|_{L_\lambda^{-k,p}(\mathbb{R}^n)}^p = \inf_{(g_\alpha)} \sum_{|\alpha| \leq k} \|g_\alpha\|_{L_\lambda^p(\mathbb{R}^n)}^p$$

where we take infimum over all collections $(g_\alpha)_{|\alpha| \leq k}$ of functions in $L_\lambda^p(\mathbb{R}^n)$ that satisfy (3.2). Note that

$$L_\lambda^{k,p}(\mathbb{R}^n) = \begin{cases} 0 & \text{for } \lambda < 0 \\ L^{k,\infty}(\mathbb{R}^n) & \text{for } \lambda = 0 \\ L^{k,p}(\mathbb{R}^n) & \text{for } \lambda \geq n. \end{cases}$$

Thus Morrey spaces are of interest only for $\lambda \in (0, n)$. For a brief introduction to Morrey spaces, see [9] Section 1 in Chapter 3.

Let \mathcal{U} be an open subset of \mathbb{R}^n . Then the Morrey space $L_\lambda^{k,p}(\mathcal{U})$, with k an integer, $p \in [1, \infty)$ and $\lambda \in \mathbb{R}$, is defined as the space of all $f \in L^{k,p}(\mathcal{U})$ such that there exists $g \in L_\lambda^{k,p}(\mathbb{R}^n)$ with $f = g|_{\mathcal{U}}$. For negative k the restriction is to be understood in the sense of distributions. It is a Banach space with norm

$$(3.4) \quad \|f\|_{L_\lambda^{k,p}(\mathcal{U})} = \inf_g \|g\|_{L_\lambda^{k,p}(\mathbb{R}^n)}$$

where we take infimum over all $g \in L_\lambda^{k,p}(\mathbb{R}^n)$ such that $f = g|_{\mathcal{U}}$.

Let X be an n -dimensional smooth compact Riemannian manifold. Then the Morrey space $L_\lambda^p(X)$, with k a positive integer and $\lambda \in \mathbb{R}$, is defined as the space of all $f \in L^p(X)$ such that the norm

$$(3.5) \quad \|f\|_{L_\lambda^p(X)}^p = \|f\|_{L^p(X)}^p + \sup_{\rho \in (0, \rho_0/2]} \sup_{x \in X} \rho^{\lambda-n} \|f\|_{L^p(B_\rho(x))}^p$$

is finite; here ρ_0 is the injectivity radius of X . This norm is invariant under isometries of X . Let E be a Euclidean vector bundle over X . Let A be a smooth metric connection on E . Then the Morrey space $L_\lambda^{k,p}(X, E)$, with k a nonnegative integer, $p \in [1, \infty)$ and $\lambda \in \mathbb{R}$, is defined as the space of all $s \in L^{k,p}(X, E)$ such that the norm

$$(3.6) \quad \|s\|_{L_{\lambda,A}^{k,p}(X,E)}^p = \sum_{j=0}^k \|(|\nabla_A^j s|)\|_{L_\lambda^p(X)}^p$$

is finite. The Morrey space $L_\lambda^{-k,p}(X, E)$, with k a positive integer, $p \in [1, \infty)$ and $\lambda \in \mathbb{R}$, is defined as the space of all $s \in L^{-k,p}(X, E)$ such that the norm

$$(3.7) \quad \|s\|_{L_{\lambda,A}^{-k,p}(X,E)}^p = \inf_{(t_j)} \sum_{j=0}^k \|(|t_j|)\|_{L_\lambda^p(X)}^p$$

is finite; here we take infimum over all (t_0, \dots, t_k) such that $t_j \in L_\lambda^p(X, (T^*X)^{\otimes j} \otimes E)$ and $s = \sum_{j=0}^k (\nabla_A^*)^j t_j$. Different smooth connections A give equivalent norms.

We can get equivalent norms for $L_\lambda^{k,p}(X, E)$ as follows. Choose an atlas of good local coordinate charts $\Phi_\nu : \mathcal{U}_\nu \rightarrow \mathbb{R}^n$ and good lifts $\Psi_\nu : E|_{\mathcal{U}_\nu} \rightarrow \mathbb{R}^n \times \mathbb{R}^N$. Here good means that the maps can be extended smoothly to neighborhoods of $\overline{\mathcal{U}_\nu}$. Then

$$(3.8) \quad \|s\|^p = \sum_\nu \|(\Psi_\nu^{-1})^* s\|_{L_\lambda^{k,p}(\Phi_\nu(\mathcal{U}_\nu), \mathbb{R}^N)}^p$$

is a norm for $L_\lambda^{k,p}(X, E)$, with $k \in \mathbb{Z}$, $p \in [1, \infty)$ and $\lambda \in \mathbb{R}$; the details are left to the reader.

The norms (3.5)–(3.7) have the advantage of being invariant under isometries of X . The norm (3.8) can be used to reduce analysis in Morrey spaces on manifolds to analysis in Morrey spaces on \mathbb{R}^n .

The following Lemma shows why Morrey spaces are useful in gauge theory for invariant connections, and possibly in other invariant situations as well.

Lemma 3.1. *Let H be a compact Lie group that acts smoothly on X , in such a way that all H -orbits have dimension $\geq n - d$, and that acts smoothly on E covering the action on X . If $s \in L^{k,p}(X, E)$, with $k \in \mathbb{Z}$ and $p \in [1, \infty)$, is H -invariant, then $s \in L_d^{k,p}(X, E)$. If A is an H -invariant connection on E , then*

$$\|s\|_{L_{d,A}^{k,p}(X)} \leq c \|s\|_{L_A^{k,p}(X)}.$$

The constant c does not depend on A .

Proof. First one shows that if \mathcal{U} is an open subset of \mathbb{R}^n , \mathcal{V} is an open subset of \mathbb{R}^n with compact closure contained in \mathcal{U} , and $f \in L^p(\mathcal{U})$ only depends on (x_1, \dots, x_d) , then $f \in L_d^p(\mathcal{V})$ and

$$\|f\|_{L_d^p(\mathcal{V})} \leq c \|f\|_{L^p(\mathcal{U})}.$$

The idea is that any ball of radius r in \mathcal{V} has the order of magnitude r^{d-n} disjoint translates in \mathcal{U} in the (x_{d+1}, \dots, x_n) directions and f has the same L^p -norm on all these balls; we leave the detailed verification to the reader.

Next consider the case in the Lemma with $k = 0$. In a neighborhood of each point on X we can find local coordinates where points with the same (x_1, \dots, x_d) lie in the same H -orbit. The Lemma then follows by applying the above estimate to these coordinate charts.

For $k > 0$ the Lemma follows by applying the case $k = 0$ to the functions $|\nabla_A^j s|$. For $k < 0$ the Lemma follows by applying the case $k = 0$ to the functions $|t_j|$, once we observe that by averaging we may take the sections t_j in (3.7) to be H -invariant. \square

Next we review embedding theorems and elliptic estimates in Morrey spaces. First we have an analogue of Hölder's inequality:

Lemma 3.2. *Multiplication gives bounded linear maps*

$$L_\lambda^p(X, E) \times L_\lambda^q(X, F) \rightarrow L_\lambda^r(X, E \otimes F)$$

for all $1 \leq p, q, r < \infty$ such that $1/p + 1/q = 1/r$.

Using the norms (3.8) we see that it suffices to show that multiplication gives bounded linear maps $L_\lambda^p(\mathbb{R}^n) \times L_\lambda^q(\mathbb{R}^n) \rightarrow L_\lambda^r(\mathbb{R}^n)$. That follows immediately from Hölder's inequality applied to \mathbb{R}^n and to the balls $B_\rho(x)$.

Next we have that the Morrey spaces $L_d^{k,p}$ in n dimensions have embedding properties analogous to those of the Sobolev spaces $L^{k,p}$ in d dimensions.

Lemma 3.3. *Let $\lambda \in (0, n)$. Then there are embeddings*

$$L_\lambda^{1,p}(X, E) \rightarrow \begin{cases} L_\lambda^{p^*}(X, E) & \text{for } p \in (1, \lambda) \\ C^{0,\alpha}(X, E) & \text{for } p \in (\lambda, \infty) \end{cases}$$

where $1/p^* = 1/p - 1/\lambda$ and $\alpha = 1 - \lambda/p$.

Again it suffices to establish the analogous embedding on \mathbb{R}^n . A weaker version of the first embedding, $L_\lambda^{1,p}(\mathbb{R}^n) \rightarrow L_\lambda^q(\mathbb{R}^n)$ for $q \in [1, p^*)$, which suffices for our purposes, was first shown by S. Campanato, [4] Theorem 1.2. The sharp embedding $L_\lambda^{1,p}(\mathbb{R}^n) \rightarrow L_\lambda^{p^*}(\mathbb{R}^n)$ is due to D.R. Adams [1] Theorem 3.2; see also [6] Theorem 2. The embedding $L_\lambda^{k,p}(\mathbb{R}^n) \rightarrow C^{0,\alpha}(\mathbb{R}^n)$ is a classical result by C.B. Morrey, [18] pp. 12–14; see also [9] Theorem 1.2 in Chapter 3 and [10] Theorem 7.19.

From Lemma 3.3 we in particular get the embeddings

$$L_3^{1,2} \rightarrow L_3^6$$

and

$$L_3^{2,2} \rightarrow L_3^{1,6} \rightarrow C^{0,1/2}.$$

By combining Lemmas 3.2 and 3.3 we get the multiplications

$$\begin{aligned} L_3^{2,2} \times L_3^{2,2} &\rightarrow L_3^{2,2} \\ L_3^{2,2} \times L_3^{1,2} &\rightarrow L_3^{1,2} \end{aligned}$$

and

$$L_3^{1,2} \times L_3^{1,2} \rightarrow L_3^3 \rightarrow L^2.$$

It follows that the group operations in \mathcal{G} , the action of \mathcal{G} on \mathcal{A} , and the Yang-Mills functional are continuous in the $L_3^{2,2}$ -topology on \mathcal{G} and $L_3^{1,2}$ -topology on \mathcal{A} .

There is also a compact embedding theorem. This follows by standard methods from [1] Theorem 3.1. For the convenience of the reader we have included a proof.

Lemma 3.4. *Let $\lambda \in (0, n)$ and $p \in (1, \lambda)$. Let $q \in [1, p^*)$ where $1/p^* = 1/p - 1/\lambda$. Then the embedding $L_\lambda^{1,p}(X, E) \rightarrow L_\lambda^q(X, E)$ is compact.*

Proof. It suffices to show that if \mathcal{U} is a bounded open subset of \mathbb{R}^n , then the embedding $L_\lambda^{1,p}(\mathcal{U}) \rightarrow L_\lambda^q(\mathcal{U})$ is compact. We may assume that $q \in [p, p^*)$. Let $\alpha = \lambda/q - \lambda/p^* \in (0, 1]$. We say that $f \in C^{0,\alpha}L_\lambda^q(\mathbb{R}^n)$ if $f \in L_\lambda^q(\mathbb{R}^n)$ and

$$\sup_{0 < |h| \leq 1} |h|^{-\alpha} \|f(\cdot + h) - f(\cdot)\|_{L_\lambda^q(\mathbb{R}^n)} < \infty.$$

This is a Banach space with norm

$$\|f\|_{C^{0,\alpha}L_\lambda^q(\mathbb{R}^n)}^q = \|f\|_{L_\lambda^q(\mathbb{R}^n)}^q + \sup_{0 < |h| \leq 1} |h|^{-\alpha q} \|f(\cdot + h) - f(\cdot)\|_{L_\lambda^q(\mathbb{R}^n)}^q.$$

If $f \in L^{1,1}(\mathbb{R}^n)$, then

$$f(x) = c_n^{-1} \sum_{j=1}^n \int_{\mathbb{R}^n} (x_j - y_j) |x - y|^{-n} \partial_j f(y) dy$$

almost everywhere, where c_n is the $(n-1)$ -dimensional measure of the unit sphere in \mathbb{R}^n . Thus

$$\begin{aligned} & |h|^{-\alpha} (f(x+h) - f(x)) \\ &= c_n^{-1} \sum_{j=1}^n \int_{\mathbb{R}^n} |h|^{-\alpha} ((x_j - y_j + h_j) |x - y + h|^{-n} - (x_j - y_j) |x - y|^{-n}) \partial_j f(y) dy \end{aligned}$$

almost everywhere. Now

$$|h|^{-\alpha} |(z_j + h_j) |z + h|^{-n} - z_j |z|^{-n}| \leq c(|z + h|^{1-n-\alpha} + |z|^{1-n-\alpha})$$

almost everywhere. To see this, note that both sides are homogeneous with respect to (z, h) of degree $1-n-\alpha$. Therefore it suffices to verify the inequality on the sphere $|z|^2 + |h|^2 = 1$, which is straightforward. It follows that

$$\begin{aligned} |h|^{-\alpha} |f(x+h) - f(x)| &\leq c \int_{\mathbb{R}^n} (|x - y + h|^{1-n-\alpha} + |x - y|^{1-n-\alpha}) |\nabla f(y)| dy \\ &= c \int_{\mathbb{R}^n} |x - y|^{1-n-\alpha} (|\nabla f(y)| + |\nabla f(y+h)|) dy \end{aligned}$$

almost everywhere. It now follows from [1] Theorem 3.1 or [6] Theorem 2 that the right hand side is bounded in $L_\lambda^q(\mathbb{R}^n)$ uniformly in h . We conclude that there is a continuous embedding $L_\lambda^{1,p}(\mathbb{R}^n) \rightarrow C^{0,\alpha}L_\lambda^q(\mathbb{R}^n)$.

We define $C^{0,\alpha}L_\lambda^q(\mathcal{U})$ by extension to \mathbb{R}^n as in (3.4). It then follows that there is a continuous embedding $L_\lambda^{1,p}(\mathcal{U}) \rightarrow C^{0,\alpha}L_\lambda^q(\mathcal{U})$. It is not hard to show that if \mathcal{U} is bounded, then the embedding $C^{0,\alpha}L_\lambda^q(\mathcal{U}) \rightarrow L_\lambda^q(\mathcal{U})$ is compact. \square

Lemma 3.5. *Let $p \in (1, \infty)$ and $k \in \mathbb{Z}$. Then any elliptic partial differential operator $L_\lambda^{k+m,p}(X, E) \rightarrow L_\lambda^{k,p}(X, F)$ of order m with smooth coefficients is a Fredholm operator.*

To prove this one has to establish interior estimates for elliptic operators on bounded open subsets of \mathbb{R}^n . That was first done by S. Campanato, [5] Theorem 10.1, in the case $p = m = 2$, which suffices for our purposes. To do the general case, one has to show that singular integral operators are bounded $L_\lambda^p(\mathbb{R}^n) \rightarrow L_\lambda^p(\mathbb{R}^n)$. That is a result due to J. Peetre [22] Theorem 1.1; see also [6] Theorem 3 and [25] Proposition 3.3.

Remark 3.6. No one has bothered to show that the theory of elliptic boundary value problems extends to Morrey spaces. However, it is clear that the classical treatment of the Neumann and Dirichlet problems for the Laplacian by even and odd reflection across the boundary carries over to Morrey spaces.

Remark 3.7. We have chosen to work with smooth connections and smooth gauge transformations. One could also complete the space of connections and the group of gauge transformations in $L_3^{1,2}$ and $L_3^{2,2}$ respectively. Then one should observe that $C_0^\infty(\mathbb{R}^n)$ is not dense in $L_\lambda^{k,p}(\mathbb{R}^n)$ for $\lambda \in (0, n)$. The closure of $C_0^\infty(\mathbb{R}^n)$ in $L_\lambda^{k,p}(\mathbb{R}^n)$ for $\lambda \in (0, n)$ is a closed subspace $C^0 L_\lambda^{k,p}(\mathbb{R}^n)$ of $L_\lambda^{k,p}(\mathbb{R}^n)$. It consists of all $f \in L_\lambda^{k,p}(\mathbb{R}^n)$ such that $f(\cdot + h) \rightarrow f(\cdot)$ in $L_\lambda^{k,p}(\mathbb{R}^n)$ as $h \rightarrow 0 \in \mathbb{R}^n$. It is of course a Banach space with the same norm as $L_\lambda^{k,p}(\mathbb{R}^n)$. By the usual techniques, one can define $C^0 L_\lambda^{k,p}(X, E)$, which then is the closure of $C^\infty(X, E)$ in $L_\lambda^{k,p}(X, E)$. Thus the completions of \mathcal{A} and \mathcal{G} would be the space of connections in $C^0 L_3^{1,2}$ and the group of gauge transformations in $C^0 L_3^{2,2}$.

§4. Invariant connections

When we stated the main theorems in §2, we considered a fixed lift $\sigma : H \times P \rightarrow P$ of the action $\rho : H \times X \rightarrow X$. However, in the proofs of these theorems we will need to consider other lifts. In particular, the gauge transforms of the lift σ will play an important role. The group \mathcal{G} acts on the set of lifts of ρ as follows:

$$(g.\sigma)(h) = g \sigma(h) g^{-1}.$$

Note that if A is invariant under σ , then $g.A$ is invariant under $g.\sigma$. Also note that

$$(4.1) \quad g.\sigma = \sigma \iff g \circ \sigma(h) = \sigma(h) \circ g \text{ for all } h \in H \iff g \in \mathcal{G}^H,$$

i.e. σ is invariant under g if and only if g is invariant under σ .

If we wish to consider other lifts of ρ , then it is natural to introduce the extended gauge group $\widehat{\mathcal{G}}$ defined as the group of ordered pairs (φ, h) such that $h \in H$ and $\varphi : P \rightarrow P$ is a bundle map that covers the isometry $\rho(h) : X \rightarrow X$. There is a short exact sequence:

$$(4.2) \quad 1 \longrightarrow \mathcal{G} \longrightarrow \widehat{\mathcal{G}} \longrightarrow H \longrightarrow 1.$$

A lift σ' of ρ is by definition the the same as a splitting $H \rightarrow \widehat{\mathcal{G}}$ of this short exact sequence.

There are compatible group actions:

$$\begin{aligned} \mathcal{G} \times \mathcal{A} &\rightarrow \mathcal{A} \\ \widehat{\mathcal{G}} \times \mathcal{A} &\rightarrow \mathcal{A} \\ H \times \mathcal{B} &\rightarrow \mathcal{B}. \end{aligned}$$

The first two actions are defined the obvious way. The third action was introduced in [8] and is defined as follows: Given $h \in H$ and $[A] \in \mathcal{B}$, we choose any bundle map φ that covers the action of h on X . Then we define $h.[A] = [\varphi.A] \in \mathcal{B}$. As A and φ are unique up to gauge transformations, $\varphi.A$ is unique up to gauge transformations. Hence $[\varphi.A]$ is a well-defined element of \mathcal{B} . We denote the fixed point set of the action $H \times \mathcal{B} \rightarrow \mathcal{B}$ by \mathcal{B}^H .

For any $A \in \mathcal{A}$ these actions have isotropy subgroups

$$\begin{aligned} \Gamma_A &= \{g \in \mathcal{G} \mid g.A = A\} \\ \widehat{\Gamma}_A &= \{(\varphi, h) \in \widehat{\mathcal{G}} \mid \varphi.A = A\} \\ \Gamma_{[A]} &= \{h \in H \mid h.[A] = [A]\}. \end{aligned}$$

These are compact Lie groups, and they form a short exact sequence

$$(4.3) \quad 1 \longrightarrow \Gamma_A \longrightarrow \widehat{\Gamma}_A \longrightarrow \Gamma_{[A]} \longrightarrow 1.$$

We have $[A] \in \mathcal{B}^H$ if and only if $\Gamma_{[A]} = H$.

If A is invariant under any lift σ' of ρ , then $[A] \in \mathcal{B}^H$. Conversely, if $[A] \in \mathcal{B}^H$, then the lifts under which A is invariant are by definition precisely the splittings $H \rightarrow \widehat{\Gamma}_A$ of the short exact sequence

$$(4.4) \quad 1 \longrightarrow \Gamma_A \longrightarrow \widehat{\Gamma}_A \longrightarrow H \longrightarrow 1.$$

Arguing as usual, but using the Morrey space estimates of §3 instead of the usual Sobolev space estimates, we see that for $\varepsilon > 0$ small enough

$$\mathcal{O}_{A_0, \varepsilon} = \{A_0 + a \in \mathcal{A} \mid d_{A_0}^* a = 0 \text{ and } \|a\|_{L_{3, A_0}^{1,2}(X, T^* X \otimes \text{Ad } P)} < \varepsilon\}$$

is a local slice through A_0 for the action of \mathcal{G} on \mathcal{A} .

Lemma 4.1. *For any $A_0 \in \mathcal{A}$ with $[A_0] \in \mathcal{B}^H$ there exists $\varepsilon > 0$ such that*

$$\widehat{\Gamma}_A \subseteq \widehat{\Gamma}_{A_0}$$

for all $A \in \mathcal{O}_{A_0, \varepsilon}$.

Proof. First I claim that the action $\widehat{\mathcal{G}} \times \mathcal{A} \rightarrow \mathcal{A}$ restricts to an action

$$\widehat{\Gamma}_{A_0} \times \mathcal{O}_{A_0, \varepsilon} \rightarrow \mathcal{O}_{A_0, \varepsilon}.$$

In fact, let $(\varphi, h) \in \widehat{\Gamma}_{A_0}$ and $A = A_0 + a \in \mathcal{O}_{A_0, \varepsilon}$. Then $\varphi.A = A_0 + \varphi.a$. The norm $\|\cdot\|_{L_{3, A_0}^{1,2}}$ is $\widehat{\Gamma}_{A_0}$ -invariant, so $\|\varphi.a\|_{L_{3, A_0}^{1,2}} = \|a\|_{L_{3, A_0}^{1,2}} < \varepsilon$. Moreover,

$$d_{A_0}^*(\varphi.a) = d_{\varphi.A_0}^*(\varphi.a) = \varphi.d_{A_0}^* a = 0.$$

The claim follows.

Let now $A \in \mathcal{O}_{A_0, \varepsilon}$ and $(\varphi, h) \in \widehat{\Gamma}_A$. We have to show that $(\varphi, h) \in \widehat{\Gamma}_{A_0}$. That $A_0 \in \mathcal{B}^H$ means that $\Gamma_{[A_0]} = H$. It then follows from the exactness of the sequence (4.3) for A_0 that there exists a bundle map ψ such that $(\psi, h) \in \widehat{\Gamma}_{A_0}$. It follows from the exactness of (4.2) that $\psi = g\varphi$ for some $g \in \mathcal{G}$. Thus

$$(g\varphi, h) \in \widehat{\Gamma}_{A_0}.$$

Hence $g\varphi.A \in \mathcal{O}_{A_0, \varepsilon}$. But $\varphi.A = A$ so $g.A \in \mathcal{O}_{A_0, \varepsilon}$. It is well-known that, for ε small enough, $A, g.A \in \mathcal{O}_{A_0, \varepsilon}$ implies $g \in \Gamma_{A_0}$. Hence

$$(g, 1) \in \widehat{\Gamma}_{A_0}.$$

We conclude that $(\varphi, h) \in \widehat{\Gamma}_{A_0}$. □

We can now show an important semicontinuity property:

Proposition 4.2. *For any $A_0 \in \mathcal{A}$ there exists $\varepsilon > 0$ such that if $A \in \mathcal{O}_{A_0, \varepsilon}$, σ' is a lift of ρ , and A is invariant under σ' , then A_0 is invariant under σ' .*

Proof. If $[A_0] \notin \mathcal{B}^H$, then there exists $\varepsilon > 0$ such that $\mathcal{O}_{A_0, \varepsilon}$ is disjoint from \mathcal{B}^H , for \mathcal{B}^H is a closed subset of \mathcal{B} . It then follows that no connection in $\mathcal{O}_{A_0, \varepsilon}$ is invariant under any lift of ρ , and the Proposition holds trivially.

If $[A_0] \in \mathcal{B}^H$, then it follows from Lemma 4.1 that, for ε small enough, $\widehat{\Gamma}_A \subseteq \widehat{\Gamma}_{A_0}$. A lift σ' that preserves A_0 is by definition the same as a continuous right inverse of the homomorphism $\widehat{\Gamma}_{A_0} \rightarrow H$. A lift σ' that preserves A is by definition a continuous right inverse of $\widehat{\Gamma}_A \rightarrow H$. The Proposition follows. □

The following Proposition is essentially due to M. Furuta, [8] Lemma 2.2; see also [3] Proposition 4.3.

Proposition 4.3. *$\pi(\mathcal{E})$ is a closed subset of \mathcal{B} .*

Proof. Assume that $A_n \in \mathcal{A}^H$ and $\pi[A_n, \sigma] = [A_n] \rightarrow [A_\infty]$ in $L_3^{1,2}$ as $n \rightarrow \infty$. This means that there exist $g_n \in \mathcal{G}$ such that $g_n.A_n \rightarrow A_\infty$ in $L_3^{1,2}$ as $n \rightarrow \infty$. We have to show that $[A_\infty] \in \pi(\mathcal{E})$.

Let ε be as in Proposition 4.2. We may assume that $g_n.A_n \in \mathcal{O}_{A_\infty, \varepsilon}$ for large n . As A_n is invariant under σ , we have that $g_n.A_n$ is invariant under $g_n.\sigma$. It then follows from Proposition 4.2 that A_∞ is invariant under $g_n.\sigma$. Hence $g_n^{-1}.A_\infty$ is invariant under σ , i.e. $g_n^{-1}.A_\infty \in \mathcal{A}^H$ for large n . Thus $[A_\infty] = [g_n^{-1}.A_\infty] = \pi[g_n^{-1}.A_\infty, \sigma]$. □

Proof of Theorem 1. Let $A_n \in \mathcal{A}$ be a Palais-Smale sequence with $[A_n] \in \mathcal{B}^H$. Then A_n is invariant under the action of $\widehat{\Gamma}_{A_n}$ on P . It follows from Lemma 3.1 that

$$\|F_{A_n}\|_{L_3^2(X, \Lambda^2 T^* X \otimes \text{Ad } P)} \leq c \|F_{A_n}\|_{L^2(X, \Lambda^2 T^* X \otimes \text{Ad } P)}$$

and

$$\|d_{A_n}^* F_{A_n}\|_{L_{3, A_n}^{-1,2}(X, T^* X \otimes \text{Ad } P)} \leq c \|d_{A_n}^* F_{A_n}\|_{L_{A_n}^{-1,2}(X, T^* X \otimes \text{Ad } P)}$$

uniformly in n . Hence

$$\|F_{A_n}\|_{L_3^2(X, \Lambda^2 T^* X \otimes \text{Ad } P)} \leq cM \quad \text{for all } n$$

and

$$\|d_{A_n}^* F_{A_n}\|_{L_{3, A_n}^{-1,2}(X, T^* X \otimes \text{Ad } P)} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

It follows that there exists a sequence of gauge transformations

$$g_n \in \mathcal{G}$$

such that $g_n.A_n$ converge in $L_3^{1,2}$ -norm to a smooth Yang-Mills connection A_∞ as $n \rightarrow \infty$. In fact, for $n = 3$, where $L_3^{1,2} = L^{1,2}$, this is well-known. It is proven exactly the same way for $n \geq 4$, using the Morrey space estimates in §3 instead of the usual Sobolev space estimates.

It follows from Proposition 4.3 that $[A_\infty] \in \pi(\mathcal{E})$. In other words, A_∞ is gauge equivalent to a connection in \mathcal{A}^H . After adjusting the gauge transformations g_n we may thus assume that $A_\infty \in \mathcal{A}^H$. \square

Proof of Theorem 2, (2) \Rightarrow (1). Assume that Condition (2) holds. Let $A_n \in \mathcal{A}^H$ be a Palais-Smale sequence. It follows from Theorem 1 that there exists a sequence $g'_n \in \mathcal{G}$ and a Yang-Mills connection $A'_\infty \in \mathcal{A}^H$ such that

$$(4.5) \quad g'_n.A_n \rightarrow A'_\infty \quad \text{in } L_3^{1,2} \quad \text{as } n \rightarrow \infty.$$

Let ε be as in Proposition 4.2. We can choose the sequence g'_n so that $g'_n.A_n \in \mathcal{O}_{A'_\infty, \varepsilon}$ for large n . We have that $g'_n.A_n$ is invariant under $g'_n.\sigma$. It then follows from Proposition 4.2 that A'_∞ is invariant under $g'_n.\sigma$ for large n . Hence $g_n'^{-1}.A_n$ is invariant under σ ; in other words

$$(4.6) \quad g_n'^{-1}.A'_\infty \in \mathcal{A}^H$$

for large n . In particular $[g_n'^{-1}.A'_\infty, \sigma] \in \pi^{-1}[A'_\infty]$ for large n . It follows from Condition (2) that $\pi^{-1}[A'_\infty]$ is finite. After passing to a subsequence we may therefore assume that all $[g_n'^{-1}.A'_\infty, \sigma]$ are equal. This means that for any integer n_0 there exists a sequence

$$g_n \in \mathcal{G}^H$$

such that $g_n g_n'^{-1}.A'_\infty = g_{n_0}'^{-1}.A'_\infty$ for all n . Then $g_{n_0}' g_n g_n'^{-1} \in \Gamma_{A'_\infty}$. The group $\Gamma_{A'_\infty}$ is compact. Hence, after passing to a subsequence,

$$(4.7) \quad g_{n_0}' g_n g_n'^{-1} \rightarrow s \quad \text{in } L_3^{2,2} \quad \text{as } n \rightarrow \infty$$

for some

$$(4.8) \quad s \in \Gamma_{A'_\infty}.$$

It follows from (4.7), (4.5), (4.8) and (4.6) that

$$g_n \cdot A_n = (g_n g'_n{}^{-1}) \cdot (g'_n \cdot A_n) \rightarrow g'_{n_0}{}^{-1} s \cdot A'_\infty = g'_{n_0}{}^{-1} A'_\infty \in \mathcal{A}^H$$

in $L_3^{1,2}$ as $n \rightarrow \infty$. Condition (1) follows with $A_\infty = g'_{n_0}{}^{-1} \cdot A'_\infty$. \square

§5. Homomorphisms of compact Lie groups

In this section we review some facts about homomorphisms of compact Lie groups that will be used in the proofs of the implication (1) \Rightarrow (2) in Theorem 2 and Theorem 3(b). Note that we do not assume the groups to be connected.

If H is a compact Lie group and K is a Lie group, then we let $\text{Hom}(H, K)$ denote the set of continuous, and hence smooth, homomorphisms $H \rightarrow K$. We will view $\text{Hom}(H, K)$ as a subset of the Banach manifold $C(H, K)$ of continuous maps $H \rightarrow K$. The group K acts on $\text{Hom}(H, K)$ and $C(H, K)$ by conjugation.

The following rigidity theorem, for compact Lie groups H , was announced and a proof was outlined by A. Nijenhuis and R.W. Richardson [20] Theorem C. The details of the proof were carried out, for compact topological groups H , by D.H. Lee [17] Theorem 2. The proof is based on an idea by A. Weil [28]. For the convenience of the reader we give a self-contained proof essentially following [20]. Although we only need the rigidity theorem for compact Lie groups H , we state and prove it for compact topological groups H as that does not require any extra work. For more information on the continuous cohomology introduced in the proof, see [7], [19], [16] and [14] Chapter 3.

Lemma 5.1. *If H is a compact topological group and K is a Lie group, then $\text{Hom}(H, K)$ is a closed discrete union of K -orbits in $C(H, K)$.*

Proof. It is clear that $\text{Hom}(H, K)$ is a closed subset of $C(H, K)$. We will now show that each K -orbit in $\text{Hom}(H, K)$ is isolated. We write 1 for the identity elements in H and K . We also write 1 for the map $H \times H \rightarrow K$ that maps $H \times H$ to 1. Then $\text{Hom}(H, K) = T^{-1}(1)$ where the map

$$T : C(H, K) \rightarrow C(H \times H, K)$$

is defined as

$$T(\sigma)(h_1, h_2) = \sigma(h_1) \sigma(h_2) \sigma(h_1 h_2)^{-1}.$$

Let $\sigma \in \text{Hom}(H, K)$. Then the K -orbit through σ is given by the range of the map

$$S : K \rightarrow C(H, K)$$

is defined as

$$S(k)(h) = k \sigma(h) k^{-1}.$$

The tangent space at σ to the K -orbit through σ is given by the range of the differential $(S_*)_1$.

We identify the tangent space at any point in the Lie group K with the Lie algebra \mathfrak{k} by right translation. If \mathfrak{X} is a compact topological space, then this gives an identification of the tangent space at any point in the Banach manifold $C(\mathfrak{X}, K)$ with the Banach space $C(\mathfrak{X}, \mathfrak{k})$. Under these identifications, the differentials

$$\begin{aligned} (S_*)_1 : \mathfrak{k} &\rightarrow C(H, \mathfrak{k}) \\ (T_*)_\sigma : C(H, \mathfrak{k}) &\rightarrow C(H \times H, \mathfrak{k}) \end{aligned}$$

are given by

$$\begin{aligned} (S_*)_1(\xi)(h) &= \xi - \text{Ad } \sigma(h) \xi \\ (T_*)_\sigma(\lambda)(h_1, h_2) &= \lambda(h_1) + \text{Ad } \sigma(h_1) \lambda(h_2) - \lambda(h_1 h_2). \end{aligned}$$

We see that $(S_*)_1 = -\delta_1$ and $(T_*)_\sigma = \delta_2$ where the linear maps

$$\delta_q : C(H^{q-1}, \mathfrak{k}) \rightarrow C(H^q, \mathfrak{k})$$

are defined as

$$\begin{aligned} \delta_q \mu(h_1, \dots, h_q) &= \text{Ad } \sigma(h_1) \mu(h_2, \dots, h_q) \\ &+ \sum_{i=1}^{q-1} (-1)^i \mu(h_1, \dots, h_i h_{i+1}, \dots, h_q) + (-1)^q \mu(h_1, \dots, h_{q-1}). \end{aligned}$$

A short calculation shows that the linear maps δ_q define a chain complex. The cohomology groups $H_{\text{cont}}^q(H, \mathfrak{k})$ of this complex are called the *continuous cohomology* groups of H with coefficients in the H -module \mathfrak{k} . Note that continuous cohomology is defined the same way as group cohomology, except that one only considers continuous cochains.

For a compact topological group H the continuous cohomology groups are essentially trivial. In fact, define maps

$$s_q : C(H^q, \mathfrak{k}) \rightarrow C(H^{q-1}, \mathfrak{k})$$

by

$$s_q \mu(h_1, \dots, h_{q-1}) = (-1)^q \int_H \mu(h_1, \dots, h_{q-1}, h) dh,$$

where dh is the normalized Haar measure on H . Then a short calculation shows that

$$\begin{cases} s_1 \delta_1 = 1 - p \\ \delta_q s_q + s_{q+1} \delta_{q+1} = 1 \quad \text{for } q \geq 1, \end{cases}$$

where $p(\xi) = \int_H \text{Ad } \sigma(h) \xi dh$. The map p is a projection of \mathfrak{k} onto the fixed point set \mathfrak{k}^H . It follows that

$$H_{\text{cont}}^q(H, \mathfrak{k}) = \begin{cases} \mathfrak{k}^H & \text{for } q = 0 \\ 0 & \text{for } q \geq 1. \end{cases}$$

We conclude that the null space of $(T_*)_\sigma$ is the tangent space of the K -orbit through σ and that $(T_*)_\sigma$ has closed range. It then follows from the implicit function theorem that the K -orbit through σ has a neighborhood in $C(H, K)$ where $T \neq 1$ away from the K -orbit itself. \square

Lemma 5.2. *If H is a semisimple compact Lie group and K is a compact Lie group, then there exist only finitely many K -conjugacy classes of homomorphisms $H \rightarrow K$.*

Proof. Let \mathfrak{h} and \mathfrak{k} be the Lie algebras of H and K . Let $\text{hom}(\mathfrak{h}, \mathfrak{k})$ be the space of linear maps $\mathfrak{h} \rightarrow \mathfrak{k}$ and $\mathfrak{hom}(\mathfrak{h}, \mathfrak{k})$ the set of Lie algebra homomorphisms $\mathfrak{h} \rightarrow \mathfrak{k}$. Since \mathfrak{h} is semisimple, any homomorphism $\mathfrak{h} \rightarrow \mathfrak{k}$ takes values in the semisimple part \mathfrak{k}_{ss} of \mathfrak{k} . The Killing form $|X|^2 = -\text{Tr}(\text{ad } X)^2$ defines norms on \mathfrak{h} and \mathfrak{k}_{ss} . This gives a norm

$$|\lambda|^2 = \sup_{0 \neq X \in \mathfrak{h}} \frac{|\lambda X|^2}{|X|^2} = \sup_{0 \neq X \in \mathfrak{h}} \frac{\text{Tr}(\text{ad}(\lambda X))^2}{\text{Tr}(\text{ad } X)^2}$$

on $\text{hom}(\mathfrak{h}, \mathfrak{k}_{\text{ss}})$. Now $\text{ad} \circ \lambda$ gives a representation of \mathfrak{h} on \mathfrak{k} . We see that $|\lambda|$ only depends on (the isomorphism class of) this representation. It follows from the classification of representations of semisimple Lie algebras that there are only finitely many representations of \mathfrak{h} of given dimension. Hence $|\lambda|$ can assume only finitely many values. In particular, $\mathfrak{hom}(\mathfrak{h}, \mathfrak{k})$ is a compact subset of $\text{hom}(\mathfrak{h}, \mathfrak{k}_{\text{ss}})$, and hence a compact subset of $\text{hom}(\mathfrak{h}, \mathfrak{k})$.

If $\sigma \in \text{Hom}(H, K)$, then $(\sigma_*)_1 \in \mathfrak{hom}(\mathfrak{h}, \mathfrak{k})$. Thus we have a uniform bound for $(\sigma_*)_1$, and hence for $(\sigma_*)_h$ for any $h \in H$. It then follows from the Arzela-Ascoli theorem that $\text{Hom}(H, K)$ is a compact subset of $C(H, K)$. Lemma 5.2 now follows from Lemma 5.1. \square

Next we consider a short exact sequence

$$(5.1) \quad 1 \longrightarrow L \longrightarrow K \xrightarrow{\tau} H \longrightarrow 1$$

of *compact* Lie groups. A smooth homomorphism $\sigma : H \rightarrow K$ which is a right inverse of τ is called a splitting of the sequence. Two splittings are said to be equivalent if they differ by conjugation by an element of L .

Lemma 5.3. *Any K -conjugacy class in $\text{Hom}(H, K)$ contains only finitely many equivalence classes of splittings of the short exact sequence (5.1).*

Proof. If $\sigma \in \text{Hom}(H, K)$ is a splitting, then the other splittings in the K -conjugacy class of σ are of the form $k\sigma k^{-1}$ with $k \in \tau^{-1}(Z(H))$. The splittings σ and $k\sigma k^{-1}$ are equivalent, i.e. they are L -conjugates, if and only if $k \in LZ(K)$. It follows that there is a 1–1 correspondence between the L -conjugacy classes of splittings in the K -conjugacy class of σ and the elements of the group $\tau^{-1}(Z(H))/LZ(K) \cong Z(H)/\tau(Z(K))$. The Lie algebras of $Z(H)$, $\tau(Z(K))$ and $Z(H)/\tau(Z(K))$ are $\mathfrak{z}(\mathfrak{h})$, $\tau_*(\mathfrak{z}(\mathfrak{k}))$ and $\mathfrak{z}(\mathfrak{h})/\tau_*(\mathfrak{z}(\mathfrak{k}))$. It follows from the structure theorem for compact Lie groups that $\mathfrak{h} = \mathfrak{z}(\mathfrak{h}) \oplus \mathfrak{h}_{\text{ss}}$ and $\mathfrak{k} = \mathfrak{z}(\mathfrak{k}) \oplus \mathfrak{k}_{\text{ss}}$. We have $\tau_*(\mathfrak{k}_{\text{ss}}) \subseteq \mathfrak{h}_{\text{ss}}$. As τ_* is surjective, we have $\tau_*(\mathfrak{z}(\mathfrak{k})) \subseteq \mathfrak{z}(\mathfrak{h})$. Hence $\tau_*(\mathfrak{k}_{\text{ss}}) = \mathfrak{h}_{\text{ss}}$ and $\tau_*(\mathfrak{z}(\mathfrak{k})) = \mathfrak{z}(\mathfrak{h})$. Hence $Z(H)/\tau(Z(K))$ is a discrete group. It is a compact group and is hence a finite group. The Lemma follows. \square

The following Lemmas are immediate consequences of Lemmas 5.1, 5.2 and 5.3. The Lie group L acts on $C(H, K)$ by conjugation.

Lemma 5.4. *The splittings of the short exact sequence (5.1) form a closed discrete union of L -orbits in $C(H, K)$.*

Lemma 5.5. *If H is semisimple, then there exist only finitely many equivalence classes of splittings of the short exact sequence (5.1).*

§6. The fibers of the map $\pi : \mathcal{E} \rightarrow \mathcal{B}^H$

The main ingredient in the proof of the reverse implication in Theorem 2 is the following Proposition:

Proposition 6.1. *For any $[A] \in \mathcal{B}^H$, $\pi^{-1}[A]$ is a closed discrete subset of \mathcal{E} .*

Proof. The set is obviously closed. We will now show that it is discrete. Any sequence in $\pi^{-1}[A]$ is of the form $[g_n.A, \sigma]$ with $g_n \in \mathcal{G}$ and

$$(6.1) \quad g_n.A \in \mathcal{A}^H.$$

Assume that $[g_n.A, \sigma] \rightarrow [A_\infty, \sigma]$ in $L_3^{1,2}$ for some $A_\infty \in \mathcal{A}^H$. This means that there exists a sequence

$$(6.2) \quad \gamma_n \in \mathcal{G}^H$$

such that

$$(6.3) \quad \gamma_n g_n.A \rightarrow A_\infty \quad \text{in } L_3^{1,2} \quad \text{as } n \rightarrow \infty.$$

We have to show that $[g_n.A, \sigma] = [A_\infty, \sigma]$ for large n .

The orbits of the action of \mathcal{G} on \mathcal{A} are closed, so A_∞ lies in the \mathcal{G} -orbit of A . Then there exists a sequence $k_n \in \mathcal{G}$ such that

$$(6.4) \quad k_n \rightarrow 1 \quad \text{in } L_3^{2,2} \quad \text{as } n \rightarrow \infty$$

and $k_n \gamma_n g_n.A = A_\infty$. It then follows from the compactness of the isotropy subgroups that, after passing to a subsequence, $k_n \gamma_n g_n \rightarrow g_\infty$ in $L_3^{2,2}$ as $n \rightarrow \infty$ for some $g_\infty \in \mathcal{G}$. It then follows from (6.4) that

$$(6.5) \quad \gamma_n g_n \rightarrow g_\infty \quad \text{in } L_3^{2,2} \quad \text{as } n \rightarrow \infty$$

and from (6.3) that

$$(6.6) \quad g_\infty.A = A_\infty.$$

By (6.1), $g_n g_\infty^{-1}.A_\infty = g_n.A \in \mathcal{A}^H$. In other words, $g_n g_\infty^{-1}.A_\infty$ is invariant under σ , so A_∞ is invariant under $g_\infty g_n^{-1}.\sigma$. A lift that preserves A_∞ is by definition the same as a continuous right inverse of the homomorphism $\widehat{\Gamma}_{A_\infty} \rightarrow H$. Thus $g_\infty g_n^{-1}.\sigma$ is a

homomorphism $H \rightarrow \widehat{\Gamma}_{A_\infty}$, in particular an element of $C(H, \widehat{\Gamma}_{A_\infty})$. By (6.2) and (4.1), $\gamma_n \cdot \sigma = \sigma$. It then follows from (6.5) that

$$g_\infty g_n^{-1} \cdot \sigma = g_\infty (\gamma_n g_n)^{-1} \cdot \sigma \rightarrow \sigma \quad \text{in } C(H, \widehat{\Gamma}_{A_\infty}) \quad \text{as } n \rightarrow \infty.$$

It follows from Lemma 5.4, applied to the short exact sequence (4.4) with $A = A_\infty$, that $g_\infty g_n^{-1} \cdot \sigma$ lies in the same Γ_{A_∞} -orbit as σ for large n . In other words, there exists a sequence

$$(6.7) \quad s_n \in \Gamma_{A_\infty}$$

such that $s_n g_\infty g_n^{-1} \cdot \sigma = \sigma$ for large n . By (4.1),

$$(6.8) \quad s_n g_\infty g_n^{-1} \in \mathcal{G}^H$$

for large n . It follows from (6.8), (6.6) and (6.7) that

$$[g_n \cdot A, \sigma] = [s_n g_\infty \cdot A, \sigma] = [s_n \cdot A_\infty, \sigma] = [A_\infty, \sigma]$$

for large n . □

Proof of Theorem 2, (1) \Rightarrow (2). Assume that Condition (2) were false. Then there would exist a Yang-Mills connection $[A] \in \mathcal{B}^H$ with infinitely many distinct preimages $[A_n, \sigma]$ in \mathcal{E} . Then A_n would be a Palais-Smale sequence. It follows from Proposition 6.1 that the sequence $[A_n, \sigma]$ would not contain any convergent subsequence. In other words, Condition (1) would be false. □

In order to state the next Proposition we need to consider all lifts σ of ρ simultaneously. Thus we choose one representative σ_i from each gauge equivalence class of lifts of ρ ; here i ranges over some set I that parametrizes the gauge equivalence classes of lifts. Each lift σ_i gives actions $H \times \mathcal{A} \rightarrow \mathcal{A}$ and $H \times \mathcal{G} \rightarrow \mathcal{G}$. We let \mathcal{A}_i^H and \mathcal{G}_i^H be the fixed point sets of these actions and we let $\mathcal{E}_i = \mathcal{A}_i^H / \mathcal{G}_i^H$. We denote the equivalence class of $A \in \mathcal{A}_i^H$ by $[A, \sigma_i] \in \mathcal{E}_i$. There are natural maps

$$\pi_i : \mathcal{E}_i \rightarrow \mathcal{B}^H.$$

Proposition 6.2. *Let $A \in \mathcal{A}$ with $[A] \in \mathcal{B}^H$. Then there is a 1–1 correspondence between the disjoint union $\coprod_{i \in I} \pi_i^{-1}[A]$ and the set of equivalence classes of splittings of the short exact sequence (4.4).*

Proof. It follows from (4.1) that there is a 1–1 correspondence

$$(6.9) \quad \coprod_{i \in I} \mathcal{E}_i \leftrightarrow \left\{ (A, \sigma') \left| \begin{array}{l} A \in \mathcal{A}, \sigma' \text{ is a lift of } \rho, \\ \text{and } A \text{ is } \sigma'\text{-invariant} \end{array} \right. \right\} / \mathcal{G}.$$

We denote the equivalence class of (A, σ') by $[A, \sigma']$. That is consistent with the notation $[A, \sigma_i]$ for the equivalence class of $A \in \mathcal{A}_i^H$ in \mathcal{E}_i . In other words, the map to the right in (6.9) is given by $[A, \sigma_i] \mapsto [A, \sigma_i]$.

Let $A \in \mathcal{A}$ with $[A] \in \mathcal{B}^H$. Under the 1–1 correspondence (6.9) the preimage of $[A]$ in $\coprod_{i \in I} \mathcal{E}_i$ corresponds to the set of all equivalence classes $[A, \sigma']$ such that A is invariant under σ' . A lift σ' that preserves A is by definition the same as a splitting of the short exact sequence (4.4). We have $[A, \sigma'] = [A, \sigma'']$ if and only if there exists $g \in \mathcal{G}$ such that $g.A = A$, i.e. $g \in \Gamma_A$, and $g.\sigma' = \sigma''$. In other words, two splittings σ' and σ'' of (4.4) give the same element of $\coprod_{i \in I} \mathcal{E}_i$ if and only if they are Γ_A -conjugates. \square

Theorem 3(a) follows from Proposition 6.2. Theorem 3(b) follows from Proposition 6.2 and Lemma 5.5.

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